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A matrix-oriented POD-DEIM algorithm applied to semilinear differential matrix equations

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We are interested in numerically approximating the solution $\boldsymbol{U}(t) \in \mathcal{S}$ to the following semilinear matrix differential equation

$$\dot{\boldsymbol{U}}(t) = \boldsymbol{A}\boldsymbol{U}(t) + \boldsymbol{U}(t)\boldsymbol{B} + \mathcal{F}(\boldsymbol{U}, t), \quad \boldsymbol{U}(0) = \boldsymbol{U}_0, \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{n_x \times n_x}, \boldsymbol{B} \in \mathbb{R}^{n_y \times n_y}$, and $t \in [0, T_f] = \mathcal{T} \subset \mathbb{R}$, equipped with appropriate boundary conditions. The function $\mathcal{F} : \mathcal{S} \times \mathcal{T} \to \mathbb{R}^{n_x \times n_y}$ is a sufficiently regular nonlinear function that can be evaluated elementwise, and $\mathcal{S}$ is a functional space containing the sought after solution.

The problem (1) arises for instance in the discretization of two-dimensional partial differential equations of the form

$$u_t = \ell(u) + f(u, t), \quad u = u(x, y, t) \quad \text{with } (x, y) \in \Omega \subset \mathbb{R}^2,\ t \in \mathcal{T}, \tag{2}$$

and given initial condition $u(x, y, 0) = u_0(x, y)$, for certain choices of the physical domain Ω. The differential operator ℓ is linear in u, typically a second order operator in the space variables, while $f : S \times \mathcal{T} \to \mathbb{R}$ is a nonlinear function, where S is an appropriate space with $u \in S$.

In this talk we present a matrix-oriented model order reduction strategy for the problem (1) that leads to a semilinear *matrix* differential equation with the same structure as (1), but of significantly reduced dimension. More precisely, we determine an approximation to $\boldsymbol{U}(t)$ of the type

$$\boldsymbol{V}_{\ell,U}\boldsymbol{Y}_k(t)\boldsymbol{W}_{r,U}^\top, \quad t \in [0, T_f], \tag{3}$$

where $\boldsymbol{V}_{\ell,U} \in \mathbb{R}^{n \times k_1}$ and $\boldsymbol{W}_{r,U} \in \mathbb{R}^{n \times k_2}$ are matrices to be determined, independent of time. Here $k_1, k_2 \ll n$ and we let $k = (k_1, k_2)$. The function $\boldsymbol{Y}_k(t)$ is determined as the numerical solution to the following *reduced* semilinear matrix differential problem

$$\begin{aligned} \dot{\boldsymbol{Y}}_k(t) &= \boldsymbol{A}_k\boldsymbol{Y}_k(t) + \boldsymbol{Y}_k(t)\boldsymbol{B}_k + \widehat{\mathcal{F}_k(\boldsymbol{Y}_k, t)} \\ \boldsymbol{Y}_k(0) &= \boldsymbol{Y}_k^{(0)} := \boldsymbol{V}_{\ell,U}^\top \boldsymbol{U}_0 \boldsymbol{W}_{r,U}, \end{aligned} \tag{4}$$

with $\boldsymbol{A}_k = \boldsymbol{V}_{\ell,U}^\top \boldsymbol{A}\boldsymbol{V}_{\ell,U}$, $\boldsymbol{B}_k = \boldsymbol{W}_{r,U}^\top \boldsymbol{B}\boldsymbol{W}_{r,U}$, and $\widehat{\mathcal{F}_k(\boldsymbol{Y}_k, t)}$ is a matrix-oriented approximation to

$$\mathcal{F}_k(\boldsymbol{Y}_k, t) = \boldsymbol{V}_{\ell,U}^\top \mathcal{F}(\boldsymbol{V}_{\ell,U}\boldsymbol{Y}_k\boldsymbol{W}_{r,U}^\top, t)\boldsymbol{W}_{r,U}. \tag{5}$$

By stacking the columns of the matrix $\boldsymbol{U}(t)$ one after the other into a long vector, a collection of existing approaches typically map the matrix-valued problem (1) to a vector-valued dynamical system, for which order reduction is a well-established procedure. Among various methods, the Proper Orthogonal Decomposition (POD) [5, 2] methodology has been widely employed. The overall effectiveness of the POD procedure is largely influenced by the capability of evaluating the nonlinear term within the reduced space, motivating a considerable amount of work towards this estimation. One very successful approach is the Discrete Empirical Interpolation Method (DEIM) [3], which is based on the Empirical Interpolation Method (EIM) originally introduced in [1].

However, a shortcoming of these vectorization procedures is the massive computational and storage demand in the offline phase. Even in the online phase, several vectors of length $N = n_x n_y$ need to be stored to lift the low-dimensional functions back to the full dimension. Here we address precisely this shortcoming, focusing on POD for dimension reduction and DEIM for interpolation of the nonlinear function.

To this end, we devise a matrix-oriented POD approach tailored towards the construction of the matrix reduced problem formulation (4). An adaptive procedure is also developed to limit the number of snapshots contributing to the generation of the approximation spaces. The reduction of the nonlinear term is then performed by means of a fully matricial interpolation using left and right projections onto two distinct reduction spaces, giving rise to a new two-sided version of DEIM. By maintaining a matrix-oriented reduction, we are able to employ first order exponential integrators at negligible costs. Numerical experiments on benchmark problems illustrate the effectiveness of the new setting [4].

References

[1] M. Barrault, Y. Maday, N. C. Nguyen and A. T. Patera, *An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations*, C. R. Math. Acad. Sci. Paris, (2004) pp. 667–672.

[2] P. Benner, V. Mehrmann and D. Sorensen, *Dimension reduction of large-scale systems*, Springer-Verlag, Berlin/Heidelberg, Germany, 2005.

[3] S. Chaturantabut and D. C. Sorensen, *Nonlinear model reduction via discrete empirical interpolation*, SIAM J. Sci. Comput., (2010) pp. 2737–2764.

[4] G. Kirsten and V. Simoncini, *A matrix-oriented POD-DEIM algorithm applied to semilinear differential matrix equations*, arXiv preprint arXiv:2006.13289, (2021).

[5] K. Kunisch and S. Volkwein, *Control of the Burgers equation by a reduced-order approach using proper orthogonal decomposition*, J. Optim. theory Appl., (1999) pp. 345–371.