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## Null space-based approaches for the least squares and symmetric saddle-point problems

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Consider saddle-point systems of the following form  $A\begin{pmatrix} u\\v \end{pmatrix} = \begin{pmatrix} A & B^T\\ B & 0 \end{pmatrix} \begin{pmatrix} u\\v \end{pmatrix} = \begin{pmatrix} f\\g \end{pmatrix}$ 

 $(1)where A \in$ 

 $\mathbb{R}^{n \times n}$  is large, sparse and symmetric positive definite and  $B \in \mathbb{R}^{k \times n} (n > k)$  has full rank. Such systems arise in various scientific applications, including the finite element approximation of PDEs in which a constraint is enforced on a subset of unknowns via a global scalar Lagrange multiplier, numerical continuation methods for large nonlinear systems of equations, constrained optimization as well as in linear least squares problems [2].

Null-space methods represent one of possible solution approaches to solve such systems. Despite these solution approaches that compute a null-space  $Z \in \mathbb{R}^{n \times (n-k)}$  basis of the undetermined block B are traditional in various engineering communities and offer advantages in some situations, general techniques to obtain such bases useful for large and sparse A have not been studied frequently in the past. There are more reasons for this. One of them may be that the resulting transformed system with the matrix  $Z^T A Z$  is often dense, or ill-conditioned, or both. Another reason may be that there is a lack of efficient preconditioners for the transformed systems even when the projected matrix  $Z^T A Z$  is reasonably sparse. The core of the talk is devoted to discussing various algorithms to find the null-space basis Z. Although in several of the approaches treatment of B that does not have full-rank is not difficult, we do not treat such cases at length.

Initial motivation for studying the null-space approaches in this talk is slightly different from the symmetric saddle-point problem given above. Consider the linear least-squares (LS) problem  $\min_x ||Ax-b||_2$ , where the system matrix  $A \in \mathbb{R}^{m \times n}$  ( $m \ge n$ ) and the right-hand side  $b \in \mathbb{R}^m$  are given. Assume that a part of the rows of A is dense and the rows have been initially permuted such that the dense rows are the last rows of A. If we assume a confor-

mal partitioning of the vector b we have  $A = \begin{pmatrix} A_s \\ A_d \end{pmatrix}$ ,  $A_s \in \mathbb{R}^{m_s \times n}$ ,  $A_d \in \mathbb{R}^{m_d \times n}$ ,  $b = \begin{pmatrix} b_s \\ b_d \end{pmatrix}$ ,  $b_s \in \mathbb{R}^{m_s}$ ,  $b_d \in \mathbb{R}^{m_d}$ , where  $m_s$  denotes the number of sparse rows of A, and  $m_d$  is the number of dense rows, with  $m = m_s + m_d$ ,  $m_s \ge n > m_d \ge 1$  and  $m_d \ll m_s$ .

The normal equations that describe the solution [3] (not representing always the best way to get the solution) become  $Cx = (C_s + A_d^T A_d)x = c, \ c = A_s^T b_s + A_d^T b_d,$ 

 $\begin{array}{c} A_s^T A_s \text{ is the reduced normal matrix. The solution of (2) can be obtained from the equivalent } (n+md) \times (n+md) \text{ system} \\ md) \text{ system} \\ \begin{pmatrix} C_s & A_d^T \\ A_d & -I \end{pmatrix} \begin{pmatrix} x \\ A_d x \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \\ \end{array}$ 

=  $m_d$ ,  $H = C_s$ ,  $B = A_d$ . But the (2, 2) block is nonzero and equal to the negative unit matrix of the appropriate dimension. Provided  $A_s$  has full column rank,  $C_s$  is symmetric positive definite. But the null-space based approach can be extended even if the (2, 2) block is nonzero and  $C_s$  only positive semidefinite. This positive definiteness is quite common in the LS problems even when the whole A is of full column rank. The approach to solve (3) also in this case for the full rank LS problem (3) is motivated by the following simple lemma.

**Lemma**: Consider  $A = A_s A_d$ ,  $As \in \mathbb{R}^{m_s \times n}$ ,  $A_d \in \mathbb{R}^{m_d \times n}$ . If A is of full column rank, then  $C_s = A_s^T A_s$  is positive definite on the null space of  $A_d$ .

Success of any null-space approach then depends on constructing a suitable null-space basis that keeps the matrix  $Z^T A Z$  sufficiently sparse and well-conditioned. Of course, it cannot be an orthogonal null-space basis computed, for example, by pivoted QR factorization of  $B^T$ . We will discuss other approaches to find the null-space basis heavily based on the fact that B is wide. That is, it has far fewer rows than columns. The approaches include computing an oblique basis Z computed from more separate QR factorizations and an approach that adds rows of B one by one and extends the basis incrementally. Another approach for constructing Z is the right oblique conjugation. Applied to  $B \in \mathbb{R}^{k \times n}$  it yields  $V \in \mathbb{R}^{n \times n}$  and lower trapezoidal  $L \in \mathbb{R}^{k \times n}$  satisfying BV =

L. (4) The null-space basis is then formed by the last n - k columns of V. This approach is directly related to Thesis of Michele Benzi [1]. Techniques implied by this Thesis have motivated the right oblique conjugation, and they may still offer further compu(2)w

tational possibilities. Not only for constructing Z but possibly also for preconditioning of the linear systems with the matrix  $Z^T A Z$ .

Our experimental results show that the null-space bases obtained from the mentioned approaches are of high quality and can push forward further research in optimization and solving constrained least squares. Linear least squares problems that contain a small number of dense rows arising from practical applications are used to illustrate our ideas and to explore their potential for solving large-scale systems. Our work was originally motivated by the paper by Howell [4]. A significant part of the obtained results has been published in [5].

## References

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