Matroid endomorphisms, derivatives, and Kolchin polynomials

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Introduction

- $\mathbb{K}$ field
- $\bar{\delta}= (\delta_1, \dots, \delta_k)$ tuple of commuting derivations on K.
- **Kolchin polynomial**: a numerical polynomial , measuring the "growth rate" of $\bar{\delta}$.
- A **multi-variate** version of the same polynomial is also known.

We can abstract from the setting of fields with derivations, and consider instead a **matroid** with a tuple $\bar{\delta}$ of commuting (quasi)-endomorphisms. In this setting too there exists a (multi-variate) Kolchin polynomial measuring the growth rate of $\bar{\delta}$.
We can then consider the situation of an o-minimal structure K with a generic tuple $\bar{\delta}$ of commuting compatible **derivations**, and use the corresponding Kolchin polynomial to give a bound to the thorn rank of $(K; \bar{\delta})$.

Joint work with E. Kaplan.

Notes

The Kolchin polynomial is defined in

E. Kolchin. *Differential algebra and algebraic groups*. Pure and Applied Mathematics, vol. 54, Academic Press, New York- London, 1973.

The multi-variate Kolchin polynomial for a tuple of derivations and field endomorphisms is in

A.B. Levin. *Multivariable difference dimension polynomials*. Sovrem. Mat. Prilozh. (2004), no. 14, Algebra, 48–70.

Contents

Matroids

A **finitary matroid** (or **pregeometry**) is given by:

- X set,
- $c\ell : \mathcal{P}(X) \to \mathcal{P}(X)$ operator;

such that $c\ell$ is a **closure operator**:

- $A \subseteq B \Rightarrow c\ell(A) \subseteq c\ell(B)$,
- $A \subseteq c\ell(A)$,
- $c\ell(c\ell(A)) = c\ell(A)$;

it satisfies the **exchange property**:

- $a \in c\ell(Bc) \setminus c\ell(B) \Rightarrow c \in c\ell(Ba)$;

it is **finitary**

- $a \in c\ell(B) \Rightarrow \exists B' \subseteq B \left(a \in c\ell(B') \,\&\, B' \text{ finite}\right)$.

Examples

- X set, $c\ell(A) = A$;
- X vector space, $c\ell(A) = span(A)$;
- X field, $c\ell =$ (field theoretic) algebraic closure;
- X geometric structure, $c\ell =$ (model theoretic) algebraic closure.

Independent sets, basis, rank, ...

$(X, c\ell)$ matroid, $A \subseteq X$.

- A is **independent** if, for all $b \in A$, $b \notin c\ell(A \setminus b)$.
- A **basis** of A is a maximal independent subset of A.
- The **rank** $r(A)$ of A is the cardinality of a basis of A (it does not depend on the choice of the basis).

- A is **independent over** C if, for all $b \in A$, $b \notin c\ell(AC \setminus b)$.
- A **basis** of A over C is a subset of A maximal among sets independent over C.
- The relative rank $r(A \mid C)$ is the cardinality of a basis of A over C

$$a \in c\ell(B) \quad \Leftrightarrow \quad r(a \mid B) = 0$$

Examples

X	$c\ell(A)$	basis of A	$r(A)$
set	A	A	$\lvert A\rvert$
vector space	$span(A)$	linear basis	$\dim(A)$
field	algebraic closure	transcendence basis	transcendence degree
geometric structure	algebraic closure		

Endomorphisms

$(X, c\ell)$ matroid.

Definition

An **endomorphism** is a map $\phi : X \to X$ such that:

$$a \in c\ell(B) \Rightarrow \phi a \in c\ell(\Phi B).$$

Equivalently:

$$r(\phi A \mid \phi B) \leq r(A \mid B).$$

Notes

Matroid endomorphisms are sometimes called **strong maps**

Examples

matroid	endomorphism
set	any map
vector space	linear map
field	field automorphism
geometric structure	bijection preserving the structure

Growth rate

Unavariate polynomial

$(X, c\ell)$ finitary matroid.
$\bar{\delta} = (\delta_1, \dots, \delta_k)$ commuting endomorphisms.
$A \subset X$ finite.

Definition

$F_A(n) = r(\bar{\delta}^{\bar{s}} A : |\bar{s}| = n)$.

Theorem

There exists a polynomial $q = q_A \in \mathbb{Q}[t]$ of degree at most $k - 1$, such that, for every $n \gg 0$,

$$F_A(n) = q_A(n)$$

Growth rate

Multivariate polynomial

$(X, c\ell)$ finitary matroid.
$\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_k)$ commuting tuples of endomorphisms of length ℓ_i.
$A \subset X$ finite.

Definition

$F_A(\bar{n}) = r(\bar{\delta}_1^{\bar{s}_1} \cdots \bar{\delta}_k^{\bar{s}_k} A : |\bar{s}_1| = n_1, \ldots, |\bar{s}_k| = n_k)$.

Theorem

There exists a polynomial $q = q_A \in \mathbb{Q}[\bar{t}]$, of degree at most $\ell_i - 1$ in t_i, such that, for every $\bar{n} \gg 0$,

$$F_A(n_1, \ldots, n_k) = q_A(n_1, \ldots, n_k)$$

Example

Hilbert polynomial

If X is a vector space, the polynomial q_A is the (multi-variate) Hilbert polynomial.

Triangular systems

Definition

A **triangular system** is a tuple of commuting maps $\delta_i : X \to X$ such that, for every n,

$$r(\delta_n A \mid \delta_{<n} A \; \delta_{\leq n} B) \leq r(A \mid B)$$

Equivalently: $a \in c\ell(B) \Rightarrow \delta_n a \in c\ell(\delta_{<n} a \; \delta_{\leq n} B)$

$\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_k)$ commuting triangular systems of length ℓ_i.
$A \subset X$ finite.
$F_A(\bar{n}) = r(\bar{\delta}_1^{\bar{r}_1} \cdots \bar{\delta}_k^{\bar{r}_k} A : |\bar{r}_1| = n_1, \ldots, |\bar{r}_k| = n_k)$.

Theorem

There exists a polynomial $q = q_A \in \mathbb{Q}[\bar{t}]$, of degree at most $\ell_i - 1$ in t_i, such that, for every $\bar{n} \gg 0$,

$$F_A(n_1, \ldots, n_k) = q_A(n_1, \ldots, n_k)$$

Quasi-endomorphisms

A quasi-endomorphism is a map $\delta : X \to X$ such that (id, δ) is a triangular system:

$$r(\delta A \mid A \, B \, \delta B) \leq r(A \mid B).$$

Equivalenyly: $a \in c\ell(B) \Rightarrow \delta a \in c\ell(B \, \delta B)$
$\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_k)$ commuting tuples of quasi-endomorphisms of length ℓ_i.
$A \subset X$ finite.

Definition

$G_A(\bar{n}) = r(\bar{\delta}_1^{\bar{s}_1} \cdots \bar{\delta}_k^{\bar{s}_k} A : |\bar{s}_1| \leq n_1, \ldots, |\bar{s}_k| \leq n_k)$.

Corollary

There exists a polynomial $p_A \in \mathbb{Q}[\bar{t}]$, of degree at most ℓ_i in t_i, such that, for every $\bar{n} \gg 0$,

$$G_A(n_1, \ldots, n_k) = p_A(n_1, \ldots, n_k)$$

Example
Kolchin polynomial

If K is a field, any **derivation** (and any field automorphism) on K is a quasi-endomorphism. The corresponding polynomial p_A is the (multi-variate) **Kolchin polynomial**.

$$p_A(\bar{n}) = r(\delta_1^{\bar{s}_1} \cdots \delta_k^{\bar{s}_k} A : |\bar{s}_1| \leq n_1, \dots, |\bar{s}_k| \leq n_k), \qquad \bar{n} \gg 0$$

Invariants
$\bar{\delta} = (\delta_1, \dots, \delta_k)$ commuting quasi-endomorphisms.
$A \subset X$ finite.

$$\bar{\delta}^\infty A := \left(\bar{\delta}^{\bar{s}} A : \bar{s} \in \mathbb{N}^k\right)$$

The leading term of p_A (univariate polynomial) does not depend on A: if

$$\bar{\delta}^\infty A = \bar{\delta}^\infty A'$$

then p_A and $p_{A'}$ have the same leading monomial.

Lemma

Let $r^{\bar{\delta}}(A)$ be the coefficient of p_A of degree k. Then, up to a multiplicative constant, $r^{\bar{\delta}}$ is the rank of a matroid on X.

$$a \in \mathrm{c}\ell^{\bar{\delta}}(B) \iff \bar{\delta}^\infty a \text{ is not } \mathrm{c}\ell\text{-independent over } B$$

Fields with generic derivations

$\mathbb{K}$ a structure expanding a field
$\bar{\delta} = (\delta_1, \dots, \delta_k)$ commuting derivations.
Assume that $(\mathbb{K}, \bar{\delta})$ is "nice".

Example

- $(\mathbb{K}, \bar{\delta}) \models DCF_0^k$
- $\mathbb{K}$ o-minimal, $\bar{\delta}$ generic tuple of commuting compatible derivations.

We can endow $(\mathbb{K}, \bar{\delta})$ with an independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\delta}$, and compute its foundation rank.

Independence relation
$\mathbb{K}$ geometric structure (e.g., $\mathbb{K} \models ACF$ or $\mathbb{K}$ o-minimal): the model-theoretic algebraic closure acl is a matroid.

Definition

$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ is the independence relation induced by the algebraic closure:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{B} C$$

if every $A' \subseteq A$ which is algebraically independent over B remains algebraically independent over BC. Define

$$A \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{B} C \qquad \Leftrightarrow \qquad \bar{\delta}^\infty A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}_{\bar{\delta}^\infty B} \bar{\delta}^\infty C$$

Lemma

If $(\mathbb{K}, \bar{\delta})$ is "nice", then $\mathop{\smile\!\!\!\!\!\!|}^{\delta}$ is a "strict" independence relation.

Thorn forking

Fact (H. Adler)

If $\perp$ is any strict independence relation, then $A \perp_C B$ implies $A \perp^{\text{þ}}_C B$.

Corollary

The foundation rank of $\perp$ is greater or equal than the thorn rank.

Remark

If $(K, \bar{\delta})$ is nice, then the foundation rank of $\perp^{\delta}$ is ω^k.

Corollary

$(K, \bar{\delta})$ is super-rosy of rank ω^k.

Notes

We assume that $\mathbb{K}$ and $(\mathbb{K}, \bar{\delta})$ have geometric elimination of imaginaries.
The results are from H. Adler's PhD thesis.

Thorn and delta rank

Question

$\perp^{\text{þ}} = \perp^{\delta}$?

Fact (H. Adler)

If $\perp$ is a "canonical" independence relation, then $\perp = \perp^{\text{þ}}$

Remark

If $\perp^{acl}$ is canonical, then $\perp^{\delta}$ is canonical.

Example

If $\mathbb{K} \models RCF$, then $\perp^{acl}$ is canonical.
[Loveys-Peterzil] There exist $\mathbb{K}$ o-minimal such that $\perp^{acl}$ is not canonical.

Notes

J. Lovey, Y. Peterzil. *Linear o-minimal structures.* Israel J. of Mathematics, 81:1–30, 1993.