Topological dynamics and measures on the space of ultrafilters

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Pisa - 29/04/2022

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Section 1

Topological dynamics and ultrafilters

Topology of ultrafilters

The set $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ is a compact Hausdorff space whose base of clopen sets is

$$\{\overline{B} := \{p \in \beta\mathbb{N} : B \in p\} : B \subseteq \mathbb{N}\}.$$

Definition

Let X be a topological space, $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $p \in \beta\mathbb{N}$. the *p-limit* $p - \lim_n x_n$ is $y \in X$ such that, for every neighborhood U of y,

$$\{n \in \mathbb{N} : x_n \in U\} \in p.$$

Theorem (Stone-Cech compactification of $\mathbb{N}$)

For every $f : \mathbb{N} \to K$ with K compact Hausdorff there exists a unique continuous $\beta f : \beta\mathbb{N} \to K$ such that $\beta f(n) = f(n)$ for every $n \in \mathbb{N}$.

Algebra of ultrafilters

If $A \subseteq \mathbb{N}$ e $n \in \mathbb{N}$,

$$A - n = \{m \in \mathbb{N} : n + m \in A\}.$$

If $p \in \beta\mathbb{N}$,

$$A - p := \{n \in \mathbb{N} : A - n \in p\}.$$

Define an associative operation $\oplus$ on $\beta\mathbb{N}$ by letting

$$A \in p \oplus q \iff A - q \in p.$$

For all $p, q \in \beta\mathbb{N}$, $p \oplus q = p - \lim_n(q - \lim_m(n + m))$.

If $p \oplus q = q \oplus p$ for every $q \in \beta\mathbb{N}$, then $p = n \in \mathbb{N}$.

Idempotents

Proposition

$(\beta\mathbb{N}, \oplus)$ is a right topological semigroup, i.e. the maps

$$\lambda_q : p \mapsto p \oplus q$$

are continuous for every $q \in \beta\mathbb{N}$.

Theorem (Ellis' Lemma)

In a compact Hausdorff right topological semigroup $(S, *)$ there are idempotents, i.e. $s \in S$ such that $s * s = s$.

Ideals

Definition

$L \subseteq \beta\mathbb{N}$ is a *left ideal* if $\beta\mathbb{N} \oplus L \subseteq L$, i.e. $p \oplus q \in L$ for every $p \in \beta\mathbb{N}$ e $q \in L$.

Proposition

In $\beta\mathbb{N}$, every left ideal contains a minimal left ideal. Moreover, in $\beta\mathbb{N}$ there is a unique minimal two sided ideal, called $K(\beta\mathbb{N})$.
$K(\beta\mathbb{N})$ is the union of all the minimal left ideals.

Topological dynamics

Definition

A *topological dynamics* is a pair (X, T) where X is a compact Hausdorff space and $T : X \to X$ is continuous.

If $x \in X$, the *orbit* of x is the set $\{T^n x : n \in \mathbb{N}\}$. The *dynamics generated* by x $X(x)$ is the closure of the orbit of x.

Define, for $p \in \beta\mathbb{N}$ ultrafilter, $T^p x := p - \lim_n T^n x$. Then

$$X(x) = \{T^p x : p \in \beta\mathbb{N}\}.$$

Topological dynamics of ultrafilters

Consider now the case $(X, T) = (\beta\mathbb{N}, S)$, where $S(q) = q \oplus 1$.

If $q \in \beta\mathbb{N}$, $S^p q = p \oplus q$ and $\beta\mathbb{N}(q)$ is the left ideal generated by q

$$\beta\mathbb{N}(q) = \beta\mathbb{N} \oplus q = \{p \oplus q : p \in \beta\mathbb{N}\} = \lambda_q[\beta\mathbb{N}].$$

For every (X, T) topological dynamics and for every $x \in X$, we have a commutative diagram

$$\begin{array}{ccc} \beta\mathbb{N} & \xrightarrow{S} & \beta\mathbb{N} \\ {\scriptstyle \tau_x}\downarrow & & \downarrow{\scriptstyle \tau_x} \\ X(x) & \xrightarrow{T} & X(x) \end{array}$$

where $\tau_x : p \mapsto T^p x$ is continuous and surjective. Moreover,

$$T^p T^q x = T^{p \oplus q} x.$$

Recurrence

Definition

In a topological dynamics (X, T), a point $x \in X$ is *recurrent* if for every neighborhood U of x the set $R_x(U) := \{n \in \mathbb{N} : T^n x \in U\}$ is nonempty.

Proposition

For $x \in X$ are equivalent:

1. *x is recurrent;*
2. *exists a $p \in \beta\mathbb{N}$ such that $T^p x = x$;*
3. *exists a $p \in \beta\mathbb{N}$ idempotent such that $T^p x = x$.*

Corollary

Recurrent ultrafilters are exactly of the form $p \oplus q$ with p idempotent.

Uniform recurrence

Definition

In a topological dynamics (X, T) a point $x \in X$ is *uniformly recurrent* if for every neighborhood U of x $R_x(U)$ is *syndetic*, i.e. there exists a $k \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, at least one among $n+1, \ldots, n+k$ is in $R_x(U)$.

Proposition

For $x \in X$ are equivalent:

1. *x is uniformly recurrent;*
2. *$(X(x), T)$ is a minimal topological dynamics;*
3. *for every $q \in \beta\mathbb{N}$ there exists $p \in \beta\mathbb{N}$ such that $T^p T^q x = x$;*
4. *there exists $p \in K(\beta\mathbb{N})$ idempotent such that $T^p x = x$.*

Corollary

The uniformly recurrent ultrafilters are exactly the ultrafilters in $K(\beta\mathbb{N})$.

Section 2

Hypernatural numbers and Loeb measures

Nonstandard extensions

Nonstandard analysis associates to every object X a new object ${}^*X$ satisfying the *Transfer principle*:

Let $P(a_1, \dots, a_n)$ be a first order property of the objects $a_1, \dots, a_n$ with only bounded quantifications. Then $P(a_1, \dots, a_n)$ if and only if $P({}^*a_1, \dots, {}^*a_n)$.

Assume that ${}^*r = r$ for every $r \in \mathbb{R}$.

Definition

Let $\xi \in {}^*\mathbb{R}$ be finite. The *standard part* $\mathrm{st}(\xi)$ of ξ is the unique $r \in \mathbb{R}$ infinitely close to ξ.

If $X \in {}^*Y$ for some Y, X is called *internal object*.

Hypernatural numbers and ultrafilters

Definition

Let $\alpha \in^* \mathbb{N}$. The *ultrafilter generated* by α is

$$\mathcal{U}_\alpha := \{A \subseteq \mathbb{N} : \alpha \in^* A\}.$$

If ${}^*\mathbb{N}$ is an $2^{\aleph_0}$-enlargement, the map $\mathfrak{u} :^* \mathbb{N} \to \beta\mathbb{N}$ defined by $\mathfrak{u}(\alpha) = \mathcal{U}_\alpha$ is surjective (non injective).

${}^*\mathbb{N}$ is topologized by taking as a clopen base the family $\{{}^*A : A \subseteq \mathbb{N}\}$. This topology is compact non Hausdorff (if $\mathfrak{u}$ is surjective).

Topological dynamics of ${}^*\mathbb{N}$

Let $s : \mathbb{N} \to \mathbb{N}$ be the succesor map. Then ${}^*s :^* \mathbb{N} \to^* \mathbb{N}$ is such that ${}^*s(\xi) = \xi + 1$.

$$\mathcal{U}_{\xi+n} = \mathcal{U}_\xi \oplus n = n \oplus \mathcal{U}_\xi .$$

We have the following commutative diagram of continuous functions:

$$\begin{array}{ccc} {}^*\mathbb{N} & \xrightarrow{\ {}^*s\ } & {}^*\mathbb{N} \\ \mathfrak{u}\big\downarrow & & \big\downarrow\mathfrak{u} \\ \beta\mathbb{N} & \xrightarrow{\ S\ } & \beta\mathbb{N} \end{array}$$

if (X, T) is a topological dynamics, $x \in X$, and $\alpha \in^* \mathbb{N}$, write $T^\alpha x := T^{\mathcal{U}_\alpha} x$.

Loeb measures

Let $I \in^* \{B \in \mathcal{P}(\mathbb{N}) : B \textit{ is finite}\}$, for instance $I = [\mu, \nu]$ for some $\mu, \nu \in^* \mathbb{N} \setminus \mathbb{N}$.

Consider the nonstandard extension of the counting map

$$|\cdot| :^* \{B \in \mathcal{P}(\mathbb{N}) : B \textit{ is finite}\} \to^* \mathbb{N}.$$

We get a finitely additive measure on the algebra of internal subsets of I:

$$\mu_I(C) := \mathrm{st}(\frac{|C|}{|I|}).$$

Assuming saturation, countable additivity is trivial and Caratheodory's Theorem applies.

Integration

Let $I \subset^* \mathbb{N}$ hyperfinite.

Definition

If $f : I \to \mathbb{R}$, an internal map $F : I \to^* \mathbb{R}$ is a *lift* of f if μ_I-a.e. $\xi \in I$ $f(\xi) = \mathrm{st}(F(\xi))$.

Proposition

$f : I \to \mathbb{R}$ is μ_I-measurable if and only if it has a lift.

Proposition

Let $F : I \to^* \mathbb{R}$ be internal and finite a.e. Then $\mathrm{st}(F)$ is μ_I-integrable and

$$\int_I \mathrm{st}(F)\, d\mu_I = \mathrm{st}\Big(\frac{1}{|I|} \sum_{\xi \in I} F(\xi)\Big).$$

Section 3

Measures on $\beta\mathbb{N}$

Pushforwards of Loeb measures

Let $I \subset^* \mathbb{N}$ be hyperfinite. Define a measure on the σ-algebra generated by clopen sets of $\beta\mathbb{N}$ by letting

$$m_I(Y) := \mu_I(\{\xi \in^* \mathbb{N} : \mathcal{U}_\xi \in Y\}).$$

Proposition

- *For every $p \in \beta\mathbb{N}$ $m_I(\{p\}) = 0$.*
- *If $I \subset^* \mathbb{N}$ is an interval, $m_I(\{p \in \beta\mathbb{N} : p = p \oplus p\}) = 0$.*
- *There exist intervals $I \subseteq \mathbb{N}$ such that $m_I(K(\beta\mathbb{N})) = 0$.*

Proposition (Lindstrom)

Assuming $(2^{\aleph_0})^+$-saturation, m_I is a Borel measure.

Proposition

The map $S : p \mapsto p \oplus 1$ is measurable and measure preserving.

Induced measures on topological dynamics

Assume $(2^{\aleph_0})^+$-saturation. Fix I hyperfinite interval and consider the measure m_I on $\beta\mathbb{N}$.

Definition

Let (X, T) be a topological dynamics and let $x \in X$. Define a measure ν_I^x on X by letting

$$\nu_I^x(V) := m_I(\{p \in \beta\mathbb{N} : T^p x \in V\}).$$

Proposition

- *ν_I^x is a Borel probability measure supported on $X(x)$;*
- *T is ν_I^x-measurable and measure preserving.*

Generic points

Definition

Let (X, ν, T) be a topological measure preserving system. A point $x \in X$ is *generic* for ν along a sequence of intervals $\Phi = \{I_n\}_{n\in\mathbb{N}}$ if for every continuous $f : X \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{|I_n|} \sum_{k\in I_n} f(T^k x) = \int_X f \, d\nu.$$

Theorem

Let (X, ν, T) be a metric measure preserving topological dynamics. Let $\Phi = \{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals. Then $x_0 \in X$ is generic for ν along Φ if and only if

$$\nu \restriction Bor(X) = \nu_{I_\eta}^{x_0} \restriction Bor(X) \text{ for every } \eta \in^* \mathbb{N} \setminus \mathbb{N}.$$

Idea of the proof

Theorem (Riesz)

Let X be locally compact Hausdorff and let ν_1, ν_2 be Radon measures on X. Then $\nu_1 = \nu_2$ on Borel sets if and only if for every $f \in \mathcal{C}(X)$ $\int_X f\, d\nu_1 = \int_X f\, d\nu_2$.

Lemma

Let $f : X \to \mathbb{R}$ be continuous and fix $x \in X$. Define $\phi : \mathbb{N} \to \mathbb{R}$ by letting $\phi(n) := f(T^n x)$. Then ${}^*\phi$ is a lift of $f \circ \tau_x \circ \mathfrak{u} : \xi \mapsto f(T^\xi x)$.

$$\int_X f\, d\nu_{I_\eta}^{x_0} = \int_{I_\eta} (f \circ \tau_{x_0} \circ \mathfrak{u})\, d\mu_{I_\eta} = \operatorname{st}\Big(\frac{1}{|I_\eta|} \sum_{\xi \in I_\eta} ({}^*\varphi(\xi))\Big)$$

$$= \mathcal{U}_\eta - \lim_n \frac{1}{|I_n|} \sum_{k \in \Phi_n} f(T^k x_0) \doteq \lim_{n \to \infty} \frac{1}{|I_n|} \sum_{k \in I_n} f(T^k x_0) \Big(= \int_X f\, d\nu\Big).$$

Motivations

Consider the Bernoulli shift $\mathrm{sh} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $\mathrm{sh}(B) := B - 1$.
Let $A \subseteq \mathbb{N}$. The dynamics generated by A in $2^{\mathbb{N}}$ is $\{A - q : q \in \beta\mathbb{N}\}$.

Assume $p \in \beta\mathbb{N}$ and $\Phi = \{I_n\}_{n\in\mathbb{N}}$ be such that

$$d_p^{\Phi}(A) := p - \lim_n \frac{|A \cap I_n|}{|I_n|} = a > 0.$$

Let $\eta \in^* \mathbb{N} \setminus \mathbb{N}$ be such that $\mathcal{U}_\eta = p$. Then the measure $\nu_{I_\eta}^A$ is a measure on $2^{\mathbb{N}}$ supported on the dynamics generated by A. If A is generic,

$$U \textit{ clopen } \Rightarrow \nu_{I_\eta}^A(U) = d_p^{\Phi}(\{n \in \mathbb{N} : A - n \in U\}).$$

E.g. if $E = \{B \in 2^{\mathbb{N}} : 1 \in B\}$ then $\nu_{I_\eta}^A(E) = d_p^{\Phi}(A - 1) = d_p^{\Phi}(A) = a$.

THANK YOU!